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AN EXPERIMENT IN CORRELATING FRESHMAN MATHEMATICS.

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Perhaps everyone would agree that it is desirable (if it is feasible) to introduce the study of calculus much earlier than is customary, to correlate it more closely with the elementary subjects and these in turn more closely with each other, and to exhibit actual uses for each topic at the time it is taught.

Quite commonly, unless a student presents trigonometry and college algebra for admission to college, he does not reach integral calculus until the middle of his sophomore year, and in some excellent colleges not until his junior year. Consequently not even the simplest integrations can be employed in the early physics courses. Moreover, students of the natural or social sciences who need a little knowledge of integral calculus must take two or three years of college mathematics in order to get it. Those who can devote but one year to mathematics in college must usually stop with no idea at all of mathematics beyond trigonometry and college algebra, and a very inadequate conception of the uses of even these subjects.

In fact, how *can* the matter of applications be adequately treated when the different branches are studied separately? Few practical problems depend for their solution upon a single branch of mathematics. Such topics as trigonometric analysis show their full value only by the closest sort of correlation.

Again, when we see 200 freshmen spend a couple of weeks decomposing the most complicated fractions into partial fractions, in order that 20 of their number may be able to use this process two years later in the integral calculus, we not only question the value of the process for the freshmen but we are inclined to doubt the law of the conservation of energy. This is an extreme case; but, wherever a student must go through college algebra, trigonometry and analytic geometry before getting any calculus, similar difficulties exist.

Of course, several institutions have partially overcome these difficulties for students of the sciences and others who wish to get a glimpse of higher mathematics, by offering survey courses designed primarily for those who do not expect to specialize in mathematics. But how about the students who *do* intend to specialize—who are expecting to work in engineering, astronomy, or physics,

or to teach mathematics? Are not these the very ones who should begin using the fine tools of the calculus as soon as possible in their other studies—the very ones, moreover, who need to acquire the sort of familiarity with the calculus that comes only from using it persistently during several years? Again, are not these specialist students the ones to whom we should be most anxious to show the relations and meaning of the various branches of mathematics all along the way; that is, the ones for whom we should correlate the subjects most carefully?

Although there is some difference of opinion upon this point, it seems to the writer that a combination course which gives a preliminary bird's-eye view of the field, will, if worked out with sufficient care, prove to be not only the best final course for non-specialist students but also the best introductory course for specialist students. To be effective such a course must not be a mere collection of parts of the several subjects; it must correlate the topics and have a reasonable degree of unity and coherence. And, to meet the needs mentioned above, it must neither presuppose trigonometry nor omit integral calculus from the work of the first year.

Possibly any feeling which may exist that the old separate courses give a better foundation for specializing in mathematics is due to the difficulty of arranging a general course so as to make adequate provision for thorough drill on the various topics. But this difficulty can be overcome. Again, it is often doubted that much calculus can be taught before college algebra, trigonometry and analytics. To be sure, we cannot teach all of the ordinary calculus, but enough of it can be taught to be very serviceable and illuminating. And the rest can be given in connection with trigonometry, algebra, and analytic geometry.

An outline of a course which has been evolved and taught for several years at Reed College will serve to show one way in which this sort of thing can be done effectively. The subjects treated are: (1) Some practical uses of graphs; (2) Some important limit concepts; (3) Differentiation; (4) Integration; (5) Trigonometric functions of acute angles; (6) Logarithms; (7) Further differentiation and integration; (8) Uses of rectangular coördinates; (9) Solution of equations; (10) Polar coördinates and trigonometric functions in general; (11) Trigonometric analysis; (12) Definite integrals; (13) Progressions and series; (14) Probability and least squares.

We start with graphs because of their simplicity, obvious utility, and natural association with the foundations of the calculus. To illustrate: the student draws a velocity-time graph from a table, and uses it for interpolating, getting average and instantaneous accelerations, and finding the total distance traveled. He is obliged to consider the questions: What is an instantaneous rate? What is the true average velocity during any interval, if the instantaneous velocity increases as in the graph? These questions are further emphasized by attempting to calculate approximately the instantaneous rate of an algebraic function and the area under its graph.

Thus he is led inevitably to the careful analysis of several familiar ideas, such as speed, rate, tangent line, mean value, length, area, and volume. And the

formulation of these concepts in terms of limits shows him both their real meaning (the understanding of which, by the way, is a thing worth while for every educated person), and also how he must proceed in order to calculate them. When he has seen these he has the basic ideas of the calculus.

In fact, before he suspects that he is dealing with anything so terrible, the student has differentiated numerous simple functions (polynomials in x and $1/x$) by the increment method, and has solved easy problems on instantaneous rates, differential corrections and maxima and minima. Of course these have to be selected with great care. After getting the formulas for differentiating positive and negative powers of x at sight he meets simple problems on motion, flexure of beams, and other rates, in which he needs to differentiate two or three times in succession; and other problems in which he needs to differentiate a power of a function.

The need of anti-differentiation soon arises in problems of motion, and also in calculating areas, work, volumes, fluid pressure, etc. No mention of definite integrals is made here; in fact, the practice of determining a constant of integration each time it arises both calls attention to its importance and helps the student later on to see the meaning of the term $-\phi(a)$ in $\int_a^b \phi'(x)dx$.

The student now meets some problems which he cannot solve completely because he does not know the anti-derivative, and some in which he cannot yet express the functional relation between the sides and angles of a triangle. He is reminded that he plotted numerous functions by using tabulated values, for which he knew no formula or mathematical expression; and is told that many sorts of expressions are used besides those he met in elementary algebra. Before he can make much headway it will be necessary to become familiar with various other functions besides simple powers, fractions, sums, etc.; and it will be helpful to consider, first, the way in which the sides and angles of a triangle are related.

The actual definition of the trigonometric functions and solutions of triangles is prefaced, however, by some little use of a protractor. With it the student solves roughly such triangles as arise in surveying or statics. This is nothing more mysterious than drawing a figure to scale and reading off the required parts. This construction work not only makes the student feel that he has a ready solution or check in any such problem, but also helps him to be clear about the application of trigonometry to these subjects. Moreover, before he has any trigonometric functions to think about, it is a good time for him to get the few principles of force-resultants, which he needs to know for the problems in statics. Any one can quickly master such problems as the graphical analysis of a simple framed structure composed of a few triangles. But, of course, we must lead him by natural steps from the parallelogram of forces to such analysis.

At this stage only four functions are introduced, and three-place values are used at first, checked by the protractor to emphasize their meaning. But larger tables are soon used which make the student feel the need of easier methods of calculating, in spite of various arithmetical devices and short cuts which are

shown him. In passing be it said that some oblique triangles are solved by dissecting them or by using the sine law and cosine law.

The work on logarithms is prefaced by use of the notation 2.417×10^{-8}, etc., so common in scientific work. This is both useful in itself and helpful in making clear the treatment of characteristics of logarithms. No rule for characteristics is needed if the student gets to thinking of every number as expressible in this "scientific notation." This saves some time and trouble, and illustrates one method by which much time is saved all through the course: viz., teaching *processes*, instead of rules which take time to learn and are likely to become mechanical. Another illustration of this is the matter of interpolation; in connection with the first graphical work the student learns to interpolate in any table by a method of proportional parts. Consequently he does not need later to be taught how to interpolate in trigonometric, logarithmic, or other numerical tables. A hint suffices to show him the short-cuts after he has had some practice in using the old general process. Of course, no base but 10 is even mentioned until all the actual computing and applications have been covered, which include formulas of physics, engineering, geometry, trigonometry, interest, etc. And it is interesting that students themselves prove the laws for any base, having seen their nature clearly while working with logarithms as exponents of 10.

The student is now ready to differentiate $\log v$ and e^v, to perform integrations involving these, and to solve problems on the "compound interest law." Here he learns also to differentiate products and quotients, which might have been taken up earlier, but at the risk of confusion. In the first calculus work the student had practically nothing in the way of formulas to think about; consequently his whole attention could be given to methods of applying the calculus, and to the meaning of the processes. Each time he learns to differentiate a new type of function, he makes applications which use all the old principles. Thus, through the year, there is frequent review of the underlying principles of rates, acceleration, calculation of areas, work, volumes, differential approximations, etc.

This is followed by further graphical work, but this time from the standpoint of coördinate geometry, while heretofore it has been based merely upon the representation of a function. Starting with the plotting of points, fixed or moving, the student sees how to study the motion of a point by its parametric equations, $x = f_1(t)$, $y = f_2(t)$; finding the velocity and acceleration at any point, and the length of the path traveled. In particular, he deals with the motion of a projectile, both when its equations are given and when he must find them by integration. Next, by seeking a sure test as to whether a given point lies on a certain line or curve, the student sees the relation between a curve and its equation. He then derives the equations of various loci, proves analytically some simple theorems in the geometry of triangles and thus comes to see that algebraic methods may be systematically employed to study geometry. This is further emphasized by the discovery of some properties of conics (previously unknown to him) from their equations which he has set up. The idea of sliding a curve helps him to recognize the locus of any quadratic which lacks the xy term. Applica-

tions of the conics are pointed out here. This work is concluded by using ordinary graph paper to discover physical laws of the linear type, and logarithmic and semi-logarithmic paper to discover some other laws.

This is followed by work on the solution of equations, some of which, by the way, was included in the first graphical work. At this stage, the student is able to use differential approximations to advantage, and the idea of sliding a curve leads him easily to Horner's method. Of course, he also learns to find rational roots exactly by trial. The geometrical interpretation of simultaneous equations leads to some remarks on the fascinating topic of n-dimensional space.

The work on polar coördinates includes the plotting of fixed and moving points with special reference to circular motion, both uniform and non-uniform. In connection with this are studied the relation of an arc to its central angle, radians, estimates involving small angles, simple harmonic motion and the differentiation of the sine and cosine. The trigonometric functions are now defined for angles of any size, their graphs are drawn and their relations to acute angles are studied. Applications are made to alternating currents, parametric equations, etc.

The work on trigonometric analysis begins with the primary relations among the functions of a single angle (which the student has never yet heard of), and the uses of these relations, together with the double angle formulas, in differentiation and integration, including the method of rationalizing an algebraic integral by a trigonometric substitution. The usual formulas are developed for the direct logarithmic solution of oblique triangles and the addition formulas are proved and applied.

The student is now able to handle some of the fairly complicated integrals which arise in applications of the integral calculus. At the same time he learns to calculate volumes by double integrations in simple cases, and to draw plane sections of simple surfaces.

This is followed by some work on progressions and series with applications to interest and annuities, computation of logarithms and trigonometric functions, and integration by expansion into series. Some remarks on imaginary logarithms and Fourier's series are usually made at this point (to the student's interest, if not to his profound understanding!).

In conclusion, some work is usually done on elementary permutations and combinations and simple problems of chance. A brief introduction to the theory of probability lets the student use the method of least squares in the simplest case.

The course as given takes four hours a week through the year. The ground could be covered by lectures in less time; but with so wide a range of topics, it is important to allow time for considerable practice in class. Some time is saved by usually taking up a new topic before leaving the old,—that is, including review problems in nearly every assignment. The closeness with which successive topics are correlated and the frequency of contact with earlier topics make it possible for students to assimilate the material pretty well in spite of the rapidity

of their progress. Concise summaries in mimeographed form help greatly in fixing the grasp of the topics, especially as no text is available which covers any large part of the course from the same standpoint.

The treatment of topics is unconventional in that it keeps their practical uses steadily to the fore, and admits only topics, some of whose uses can be exhibited when the topic is studied. This gives students constant practice in applying their mathematics and in analyzing problems. It also creates considerable enthusiasm for the subject. Students see something of the meaning of mathematical work, and they do a very large amount of work very cheerfully—one may almost say eagerly. (Approximately one thousand exercises are worked during the year, many of which are fairly substantial problems.)

A word concerning the courses which follow. Naturally they must be modified somewhat. The second year is devoted to a systematic course in calculus, but several of the usual preliminaries, which were omitted from the first course, are treated incidentally as needed,—*e. g.*, elements of solid analytics before taking up multiple integration. Also, trigonometric analysis is reviewed in detail before differentiating trigonometric functions. Since the students are already familiar with the elements and general principles, they can be given an excellent grasp of the subject in this year's work. Besides covering a very full text-book treatment, they have time for some use of imaginaries in trigonometric reductions, for elementary problems on Fourier's series and calculus of variations, and for some practice in formulating as well as solving practical differential equations.

The more theoretical parts of college algebra and analytic geometry are postponed to the junior year when they can be dealt with adequately in connection with modern developments in algebra and geometry. Thus it is possible to devote the junior and senior years to mathematics of a fairly advanced type.

It seems to the writer that it would be perfectly feasible to give the freshman course outlined above in the fourth year of high schools and academies wherever there are strong teachers who know the subject well. Even then students would not be starting the study of these topics as young here as abroad. It is not hard to think of advantages which would result from their earlier introduction here.

LINKAGES.

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The subject of linkages is somewhat removed from the main lines of mathematics, but although it may not be of much importance in their development, it has considerable intrinsic interest. The problem that gave rise to the study of link-motions, that of making a point move in a straight line, is of significance theoretically in furnishing a method of drawing a straight line without begging the question as we do when we copy by means of a ruler a line already made.[1]

[1] See A. B. Kempe, *How to Draw a Straight Line*, London, Macmillan, 1877, which also gives a very clear and simple account of linkages in general. It is perhaps the best introduction to the subject, and its footnotes give references to the original papers up to its date.